MAT 1847 - Holomorphic Dynamics

Lecture 4

Riemann Surfaces

$\hat{\mathbb{C}}$

$\mathbb{C} \longrightarrow \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*, \quad \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$

$\mathbb{H} \longrightarrow \mathbb{H}/_{z\mapsto\lambda z} \simeq A_R = \{1 < |z| < R\}$

$\mathbb{H}/\mathbb{Z} \simeq \mathbb{D}^*$

$\Gamma < PSL_2(\mathbb{R})$ discrete, torsion free

$\mathbb{H}/\Gamma$ e.g. $\mathbb{H}/\Gamma(2) \simeq \hat{\mathbb{C}}\setminus\{0,1,\infty\}$

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{l} a,b,c,d \in \mathbb{Z} \\ ad-bc=1 \end{array} \quad \begin{array}{l} a,d \equiv 1\ (2) \\ b,c \equiv 0\ (2) \end{array} \right\}$$

Every hyperbolic surface carries a Poincaré metric

E.g.: if $U \subseteq \hat{\mathbb{C}}$ open connected "and" $\#|\hat{\mathbb{C}} \setminus \mathcal{U}| \geq 3$ then $\mathcal{U}$ is hyperbolic.

Thm (Schwarz-Pick-Ahlfors)

Any holo map $f: S \to T$ between hyperbolic R.S. which is not constant satisfies

$$\rho_T(f(x), f(y)) \leq \rho_S(x, y)$$

$\uparrow$

Poincaré metric on T

and if equality holds, then f is a local isometry.

Pf

$$\begin{tikzcd} \mathbb{D} \arrow[r, "\tilde{f}"] \arrow[d] & \mathbb{D} \arrow[d] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

Apply Schwarz lemma to $\tilde{f}$.

E.g.: $f: \mathbb{D}^* \longrightarrow \mathbb{D}^*$

$f(z) = z^2$

however

$$\begin{tikzcd} \mathbb{H} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{H} \arrow[d] \\ \mathbb{D}^* \arrow[r, "f"] & \mathbb{D}^* \end{tikzcd}$$

$\pi(z) = e^{2\pi i z}$

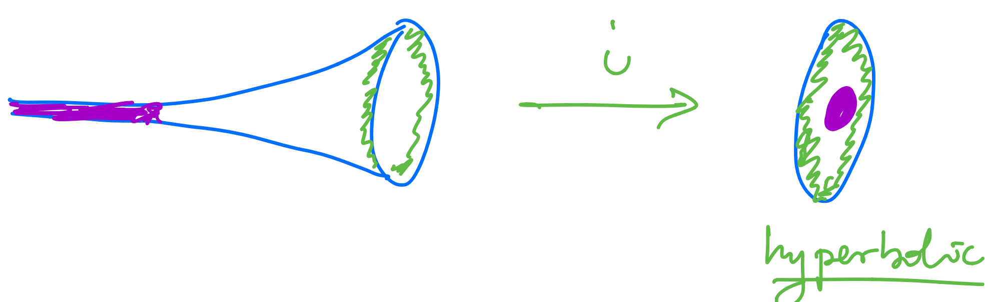
$$e^{4\pi i z} = f\left(e^{2\pi i z}\right) = e^{2\pi i \tilde{f}(z)}$$

$\tilde{f}(z) = 2z$ is an isometry $\mathbb{H} \to \mathbb{H}$

Note : $f : \mathbb{D} \to \mathbb{D}$ is NOT an isometry (it contracts strictly near 0)

$$\mathbb{D}^* \hookrightarrow \mathbb{D}$$

$$\rho_{\mathbb{D}}(z, w) < \rho_{\mathbb{D}^*}(z, w)$$



Thm (Picard)
Any holo map from $\mathbb{C}$ to $\mathbb{C}$ which omit three different values is constant.

Pf.

$$\begin{tikzcd} & \mathbb{D} \arrow[d, "\pi"] \\ \mathbb{C} \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & \mathbb{C} \setminus \{a, b, c\} \end{tikzcd}$$

$\tilde{f}$ must be constant by Liouville.

Normal Families

Def.: Let S', S be R.S., with S compact. Then a collection $\mathcal{F}$ of holo $f : S' \to S$ is NORMAL if every sequence of elts of $\mathcal{F}$ has a subsequence which converges locally uniformly.

E.g.: $f_n(z) = z^n$ $\quad (f_n) : \mathbb{D} \to \mathbb{D}$ Yes

$f_n(z) \to 0$ for all $z \in \mathbb{D}$.

$\forall r < 1$, if $|z| \leq r$ then

$|z^n| \leq r^n \to 0$ uniformly

$(f_n)(z) := z^n \quad (f_n) : \mathbb{C} \to \mathbb{C}$ No

Limit is NOT uniform

$f_n(z) \to 0$ if $|z| < 1$

$f_n(z) \to \infty$ if $|z| > 1$.

Def': If S is not compact $\mathcal{F} = \{f : S' \to S \text{ holo}\}$ is normal if every infinite sequence has a subsequence which either converges locally uniformly, or images of a compact set eventually miss any compact subset of S.

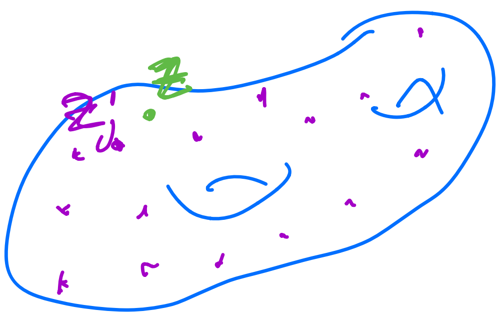
E.g. 1 $f_n(z) := z^n$ $f_n : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \overline{\mathbb{D}}$ is normal

Montel's theorem

Any collection of holo functions from S to a hyperbolic subset $\mathcal{U}$ of $\hat{\mathbb{C}}$ is NORMAL.

Pf:

$\boxed{0}$ There is a dense ctble subset $\{z_j\} \subseteq S$ and a subsequence $(f_n) \subset \mathcal{F}$ s.t. $f_n(z_j)$ converges as $n \to \infty$, for any j, to a point in $\overline{\mathcal{U}}$.



$$\lim_{n\to\infty} f_n(z_j) =: g(z_j) \in \overline{\mathcal{U}}$$

$\boxed{1}$ if $g(z_j) \in \mathcal{U}$ for all j.

Pick ε, $z \in \mathcal{U}$, then $\exists\, j$ s.t.

$$\rho_{\mathcal{U}}(z, z_j) < \varepsilon$$

$$\rho_{\mathcal{U}}(f_m(z), f_n(z)) \leq$$

$$\leq \rho_{\mathcal{U}}(f_m(z), f_m(z_j)) + \rho_{\mathcal{U}}(f_m(z_j), f_n(z_j)) + \rho_{\mathcal{U}}(f_n(z_j), f_n(z))$$

Schwarz

$$\leq \rho_{\mathcal{U}}(z, z_j) + \rho_{\mathcal{U}}(f_m(z_j), f_n(z_j)) +$$

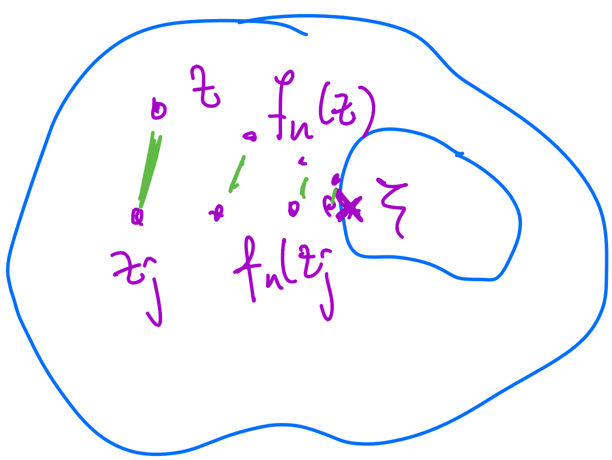
$$+\rho_{\mathcal{U}}(z_j, z)\Big)$$

$$\leq \varepsilon + \varepsilon + \varepsilon$$

$\boxed{2}$ if $\exists\, j$: $f_n(z_j) \longrightarrow g(z_j) \notin \mathcal{U}$

<u>Hence</u>: for $z \in \mathcal{U} \cap B(z_j, R)$

$$\rho_{\mathcal{U}}(f_n(z_j), f_n(z)) \leq \rho_{\mathcal{U}}(z, z_j)$$
$$\leq R$$



Hence, since $f_n(z_j) \longrightarrow \partial \mathcal{U}$,

$$\rho_{sph}\left(f_n(z), f_n(z_j)\right) \longrightarrow 0$$

As a consequence:

$$\lim_{n\to\infty} f_n(z) = \xi \in \partial U$$

locally uniformly.